

Random patterns generated by random permutations of natural numbers

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Abstract

We survey recent results on some one- and two-dimensional patterns generated by random permutations of natural numbers. In the first part, we discuss properties of random walks, evolving on a one-dimensional regular lattice in discrete time n , whose moves to the right or to the left are induced by the rise-and-descent sequence associated with a given random permutation. We determine exactly the probability of finding the trajectory of such a permutation-generated random walk at site X at time n , obtain the probability measure of different excursions and define the asymptotic distribution of the number of "U-turns" of the trajectories - permutation "peaks" and "through". In the second part, we focus on some statistical properties of surfaces obtained by randomly placing natural numbers $1, 2, 3, \dots, L$ on sites of a 1d or 2d square lattices containing L sites. We calculate the distribution function of the number of local "peaks" - sites the number at which is larger than the numbers appearing at nearest-neighboring sites - and discuss some surprising collective behavior emerging in this model.

1 INTRODUCTION

Properties of random or patterns-avoiding permutations of series of natural numbers have been analyzed by mathematicians for many years. Studies of several problems emerging within this context, such as, e.g., the celebrated Ulam's longest increasing subsequence problem (see, e.g., Refs. [1, 2, 3] and references therein), provided an entry to a rich and diverse circle of mathematical ideas [4], and were also found relevant to certain physical processes, including random surface growth [5, 6, 7, 8] or 2D quantum gravity (see, e.g., Ref. [3]).

In this paper we focus on random permutations from a different viewpoint addressing the following question: what are statistical properties of patterns, e.g., random walks or surfaces, when random permutations are used as their generator? We note that such a generator is different of those conventionally used, since here a finite amount of numbers is being shuffled and moreover, neither of any two numbers in each permutation may be equal to each other; this incurs, of course, some correlations in the produced sequences of random numbers.

In the first part of this paper, we consider a simple model of a permutation-generated random walk (PGRW), first proposed and solved in Ref. [9]. In this model, random walk evolves in discrete time on a one-dimensional lattice of integers, and the moves of the walker to the right or to the left are prescribed by the rise-and-descent sequence characterizing each given permutation $\pi = \{\pi_1, \pi_2, \pi_3, \dots, \pi_l, \dots, \pi_{n+1}\}$. In a standard notation, the "rises" (the "descents") of the permutation π are such values of l for which $\pi_l < \pi_{l+1}$ ($\pi_l > \pi_{l+1}$).

We determine exactly several characteristic properties of such a random walk, including the probability $\mathcal{P}_n(X)$ of finding the end-point X_n of the PGRW trajectory at site X , its moments, the probability measure of different excursions and the asymptotic distribution of the number of the "U-turns" of the trajectories, which, in the permutation language, corresponds to the number of "peaks" and "through" in a given permutations.

In the second part, we focus on some statistical properties of surfaces created by randomly distributing numbers $1, 2, 3, \dots, L$ on sites of one- or two-dimensional lattices containing L sites. A number appearing at the site j determines the local height of the surface. Denoting as local surface "peaks" such sites the number at which exceeds the numbers appearing at neighboring sites, we aim to calculate the probability $P(M, L)$ of having M peaks on a lattice containing L sites.

In one dimension this can be done exactly [10]. We are also able to calculate the "correlation function" $p(l)$ defining the conditional probability that two peaks are separated by the interval l under the condition that the interval l does not contain other peaks. In 2D, determining exactly first three cumulants of $P(M, L)$, we define its asymptotic form using expansion in the Edgeworth series [11] and show that it converges to a normal distribution as $L \rightarrow \infty$ [10]. For 2D model, we will also discuss some surprising cooperative behavior of peaks.

2 PERMUTATION-GENERATED RANDOM WALK

Suppose there are two players - A and B going to play the following mindless card game. They first agree on the value of each card linearly ordering them, e.g., by putting suits in the bridge-bidding order: Clubs < Diamonds < Hearts < Spades, and adopting a convention that in a series of cards with the same suit "two" is less than "three", "three" is less than "four" and etc, while the ace has the largest value. So, in such a way, a deck of cards is labeled $1, 2, 3, \dots, 52$. Then, the deck is shuffled, the upper card is turned its face up, and our players start the game: the second card is turned face up; if its value is higher than the value of the first card, the player A receives some unit of money from the player B ; if, on contrary, its value is less than the value of the first card, player B receives a unit of money from the player A . At the next step, the third card is turned its face up and its value is compared against the value of the second; if, again, its value is higher than the value of the second card, the player A receives money from the player B , otherwise, the player A pays the player B . The process continues until the deck is over. One is curious, as usual, about the winner and the amount of his gain.

Let us now look on such a random process more formally. Let $\pi = [\pi_1, \pi_2, \pi_3, \dots, \pi_{n+1}]$ denote a random permutation of $[n + 1]$. We rewrite it next in two-line notation as

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n+1 \\ \pi_1 & \pi_2 & \pi_3 & \dots & \pi_{n+1} \end{pmatrix} \quad (2.1)$$

and suppose that this table assigns a discrete "time" variable l ($l = 1, 2, 3, \dots, n+1$, upper line in the table) to each permutation in the second line. We call, in a standard notation, as a "rise" (or as a "descent") of the permutation π , such values of l for which $\pi_l < \pi_{l+1}$ ($\pi_l > \pi_{l+1}$).

Consider now a 1d lattice, place a walker at the origin at time moment $l = 0$, and let it move according to the following rules:

-at time moment $l = 1$ the walker is moved one step to the right if $\pi_1 < \pi_2$, i.e., $l = 1$ is a rise, or to the left if $\pi_1 > \pi_2$, i.e $l = 1$ is a descent.

-at $l = 2$ the walker is moved to the right (left) if $\pi_2 < \pi_3$ ($\pi_2 > \pi_3$, resp.) and etc.

Repeated l times, this results in a random trajectory $X_l^{(n)}$, ($l = 1, 2, \dots, n$), such that

$$X_l^{(n)} = \sum_{k=1}^l s_k, \quad s_k = \text{sgn}(\pi_{k+1} - \pi_k), \quad \text{sgn}(x) \equiv \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{otherwise.} \end{cases} \quad (2.2)$$

Evidently, $X_l^{(n)}$ is just an amount of money the player A won (or lost, if $X_l^{(n)} < 0$) up to the moment when the l -th card is opened. Below we answer on a set of questions on various statistical properties of random variable $X_l^{(n)}$.

2.1 Probability Distribution of The End-Point of Trajectory $X_l^{(n)}$.

Let $\mathcal{P}_n(X)$ denote the probability that the walker is at site X at time moment $l = n$. Let \mathcal{N}_\uparrow (\mathcal{N}_\downarrow) be the number of "rises" ("descents") in a given permutation π . Evidently, the end-point X_n of the walker's trajectory is $X_n = \mathcal{N}_\uparrow - \mathcal{N}_\downarrow$. Since $\mathcal{N}_\uparrow + \mathcal{N}_\downarrow = n$, we have that $X_n = 2\mathcal{N}_\uparrow - n$ and hence, X_n is fixed by the number of rises in this permutation.

Number of permutations of $[n+1]$ with *exactly* \mathcal{N}_\uparrow rises is given by the Eulerian number [12]:

$$\left\langle \begin{matrix} n+1 \\ \mathcal{N}_\uparrow \end{matrix} \right\rangle = \sum_{r=0}^{\mathcal{N}_\uparrow+1} (-1)^r \binom{n+2}{r} (\mathcal{N}_\uparrow + 1 - r)^{n+1}, \quad \binom{n+2}{r} = \frac{(n+2)!}{r!(n+2-r)!}. \quad (2.3)$$

Consequently, $\mathcal{P}_n(X)$ is given explicitly by [9]

$$\mathcal{P}_n(X) = \frac{[1 + (-1)^{X+n}]}{2(n+1)!} \left\langle \begin{matrix} n+1 \\ \frac{X+n}{2} \end{matrix} \right\rangle. \quad (2.4)$$

Several useful integral representations of the distribution function $\mathcal{P}_n(X)$ and of the corresponding lattice Green function were also derived [9]. Expression in Eq.(2.17) may be cast into the following form:

$$\mathcal{P}_n(X) = \frac{[1 + (-1)^{X+n}]}{\pi} \int_0^\infty \left(\frac{\sin(k)}{k} \right)^{n+2} \cos(Xk) dk. \quad (2.5)$$

Note that it looks *almost* like usually encountered forms for Pólya walks [13], just the upper limit of integration is infinity but not " π "; as a matter of fact, an integral representation with the integration extending over the first Brillouin zone *only* obeys

$$\mathcal{P}_n(X) = \frac{(-1)^{n+1}}{(n+1)!\pi} \int_0^\pi \left(\sin^{n+2}(k) \frac{d^{n+1}}{dk^{n+1}} \cot(k) \right) \cos(Xk) dk, \quad (2.6)$$

and has a very different structure compared to that of Pólya walks. So does the lattice Green function $\mathcal{G}(X, z)$, associated with the result in Eq.(2.6):

$$\mathcal{G}(X, z) = \sum_{n=0}^\infty \mathcal{P}_n(X) z^n = \frac{1}{\pi z} \int_0^\pi \frac{\sin(z \sin(k)) \cos(Xk)}{\sin(k - z \sin(k))} dk. \quad (2.7)$$

To establish the asymptotic form of $\mathcal{P}_n(X)$ we go to the limit $z \rightarrow 1^-$, and, inverting the expression in Eq.(2.7) in this limit, we find that $\mathcal{P}_n(X)$ converges to a *normal* distribution:

$$\mathcal{P}_n(X) \sim \left(\frac{3}{2\pi n}\right)^{1/2} \exp\left(-\frac{3X^2}{2n}\right), \quad n \rightarrow \infty. \quad (2.8)$$

Consequently, in this limit the correlations in the generator of the walk - random permutations of $[n+1]$, appear to be marginally important; that is, they do not result in *anomalous* diffusion, but merely affect the "diffusion coefficient" making it three times smaller than the diffusion coefficient of the standard 1D Pólya walk.

2.2 Correlations in PGRW Trajectories.

So, we have a symmetric random walk, which makes a move of unit length at each moment of time with probability 1, but nonetheless loses somehow two thirds of the diffusion coefficient. To elucidate this puzzling question, we address now an "inverse" problem: given the distribution $\mathcal{P}_n(X)$, we aim to determine two- and four-point correlations in the rise-and-descent sequences and consequently, in the PGRW trajectories.

For the Fourier-transformed $\mathcal{P}_n(X)$ we find

$$\tilde{\mathcal{P}}_n(k) = \sum_{X=-n}^n \exp(ikX) \mathcal{P}_n(X) = \frac{(2i)^{n+2} \sin^{n+2}(k)}{(n+1)!} \text{Li}_{-n-1}(\exp(-2ik)), \quad (2.9)$$

$\text{Li}_{-n-1}(y)$ being a poly-logarithm function. Eq.(2.9) yields

$$\langle X_n^2 \rangle \equiv \frac{1}{3}(n+2-2\delta_{n,0}) \quad \text{and} \quad \langle X_n^4 \rangle \equiv \frac{1}{15}((5n+8)(n+2)-16\delta_{n,0}-24\delta_{n,1}+8\delta_{n,2}), \quad (2.10)$$

where $\delta_{i,j}$ is the Kronecker delta, such that their generating functions are

$$\begin{aligned} \mathcal{X}^{(2)}(z) &= \sum_{n=0}^{\infty} \langle X_n^2 \rangle z^n = \frac{z(3-2z)}{3(1-z)^2}, \\ \mathcal{X}^{(4)}(z) &= \sum_{n=0}^{\infty} \langle X_n^4 \rangle z^n = \frac{z(35z+15-80z^2+48z^3-8z^4)}{15(1-z)^3}. \end{aligned} \quad (2.11)$$

On the other hand, the second and the fourth moments of the walker's displacement obey

$$\begin{aligned} \langle X_n^2 \rangle &= \langle (\mathcal{N}_\uparrow - \mathcal{N}_\downarrow)^2 \rangle = \left\langle \left[\sum_{l=1}^n s_l \right]^2 \right\rangle = n + 2 \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^n \mathcal{C}_{j_1, j_2}^{(2)}, \\ \langle X_n^4 \rangle &= n + \frac{n(n-1)}{2} \theta(n-2) + (8(n-1)\theta(n-2) + 12(n(n-3)+2)\theta(n-3)) \mathcal{C}^{(2)}(1) + \\ &+ 24 \sum_{m=1}^{n-3} (n-2-m) \mathcal{C}^{(4)}(m) \theta(n-4), \quad \theta(x) \equiv \begin{cases} +1 & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.12)$$

where $\mathcal{C}^{(2)}(m) = \mathcal{C}_{j_1, j_{m+1}}^{(2)} = \langle s_{j_1} s_{j_{m+1}} \rangle$ is the two-point correlation function of the rise-and-descent sequence, while $\mathcal{C}^{(4)}(m) = \langle s_{j_1} s_{j_1+1} s_{j_1+m+1} s_{j_1+m+2} \rangle$ defines the four-point correlation function. Note that k -point correlations with k - odd vanish [9] and thus $\mathcal{C}^{(4)}(m)$ is an only non-vanishing form of four-point correlations.

Multiplying both sides of Eqs.(2.12) by z^n and performing summation, we find that

$$\begin{aligned}\mathcal{C}^{(2)}(z) &= \sum_{m=1}^{\infty} z^m \mathcal{C}^{(2)}(m) = \frac{(1-z)^2}{2z} \mathcal{X}^{(2)}(z) - \frac{1}{2} = -\frac{z}{3}, \\ \mathcal{C}^{(4)}(z) &= \sum_{m=1}^{\infty} z^m \mathcal{C}^{(4)}(m) = \frac{(1-z)^2}{24z^3} \mathcal{X}^{(4)}(z) + \frac{16z^2 - 7z - 3}{72z^2(1-z)} = \\ &= \frac{2}{15}z + \frac{1}{9}(z^2 + z^3 + z^4 + \dots),\end{aligned}\quad (2.13)$$

which imply

$$\mathcal{C}^{(2)}(m) = \begin{cases} -1/3 & m=1 \\ 0 & m \geq 2 \end{cases} \quad \mathcal{C}^{(4)}(m) = \begin{cases} 2/15 & \text{if } m=1 \\ 1/9 & \text{if } m \geq 2 \end{cases} \quad (2.14)$$

Equations (2.14) signify that for $m \geq 2$ the two-point correlations $\mathcal{C}^{(2)}(m = |j_2 - j_1|)$ in the rise-and-descent sequences decouple into the product $\langle s_{j_1} \rangle \langle s_{j_2} \rangle$ and hence, vanish. Consequently, two-point correlations extend to the nearest neighbors only. The four-point correlations decouple $\mathcal{C}^{(4)}(m) = \mathcal{C}^{(2)}(1)\mathcal{C}^{(2)}(1)$ for $m \geq 2$.

Equations (2.14) enable us to calculate explicitly the probabilities of different configurations of two, three and four rises and descents, which are listed below

$$p_{\uparrow, \uparrow}(m) = p_{\downarrow, \downarrow}(m) = \begin{cases} 1/6 & m=1 \\ 1/4 & m \geq 2 \end{cases} \quad p_{\uparrow, \downarrow}(m) = p_{\downarrow, \uparrow}(m) = \begin{cases} 1/3 & m=1 \\ 1/4 & m \geq 2 \end{cases}$$

Hence, configurations with two neighboring rises or two neighboring descents have a lower probability than mixed rise-descent or descent-rise sequences; in other words, rises "repel" each other and "attract" descents. Further on, we have:

$$p_{\uparrow \uparrow, \downarrow}(m) = \begin{cases} 1/4 & m=1 \\ 1/12 & m \geq 2 \end{cases} \quad p_{\uparrow \uparrow, \uparrow}(m) = \begin{cases} 1/24 & m=1 \\ \frac{1}{12} & m \geq 2 \end{cases} \quad p_{\uparrow \uparrow, \uparrow \uparrow}(m) = \begin{cases} 1/120 & m=1 \\ 1/36 & m \geq 2 \end{cases}$$

$$p_{\uparrow \uparrow, \downarrow \uparrow}(m) = p_{\uparrow \downarrow, \uparrow \uparrow}(m) = \begin{cases} 3/40 & m=1 \\ 1/18 & m \geq 2 \end{cases} \quad p_{\uparrow \uparrow, \uparrow \downarrow}(m) = p_{\downarrow \uparrow, \uparrow \uparrow}(m) = \begin{cases} 1/30 & m=1 \\ 1/18 & m \geq 2 \end{cases}$$

$$p_{\uparrow \downarrow, \uparrow \downarrow}(m) = \begin{cases} 2/15 & m=1 \\ 1/9 & m \geq 2 \end{cases} \quad p_{\uparrow \downarrow, \downarrow \downarrow}(m) = \begin{cases} 1/20 & m=1 \\ 1/36 & m \geq 2 \end{cases} \quad p_{\downarrow \uparrow, \uparrow \downarrow}(m) = \begin{cases} 11/120 & m=1 \\ 1/9 & m \geq 2 \end{cases}$$

Analyzing these results, we notice that the probabilities of different rise-and-descent sequences depend not only on the number of rises or descents, but also on their order within the sequence. Correspondingly, reconstructing the PGRW trajectories $X_l^{(n)}$ with $n = 4$ (see Fig.1) we notice that the PGRW transition probabilities depend not only on the number of steps to the right or to the left, which the walker has already made, but also on their order. In other words, the PGRW represents a genuine non-Markovian process with a memory.

Finally, for k -point correlation functions $\mathcal{C}_{j_1, \dots, j_k}^{(k)} = \langle s_{j_1} s_{j_2} s_{j_3} \dots s_{j_k} \rangle$ of the rise-and-descent sequences we find the following behavior [9]: $\mathcal{C}_{j_1, \dots, j_k}^{(k)}$ factorizes automatically into a product of the corresponding correlation functions of the consecutive subsequences, in which all j_k differ by unity, as soon as any of the distances $j_{k+1} - j_k$ exceeds unity. On the other hand, the

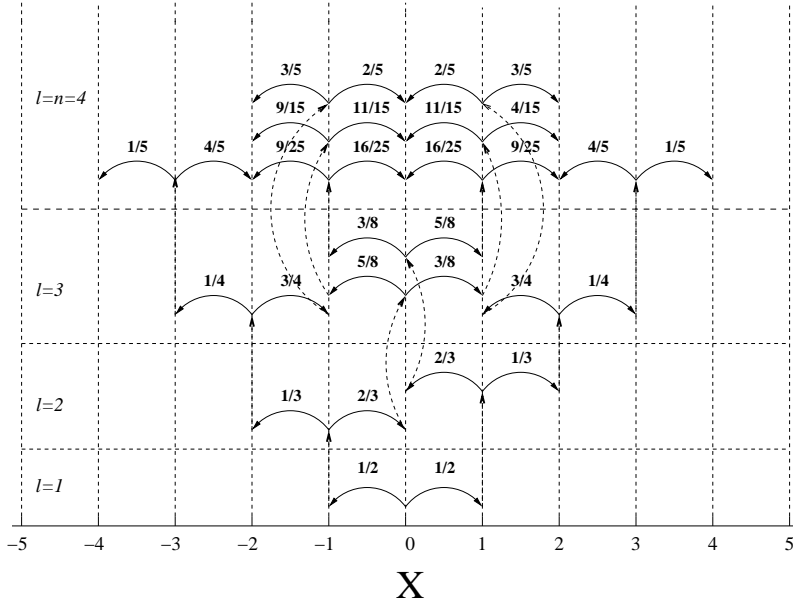


Fig. 1. A set of all possible PGRW trajectories $X_l^{(n)}$ for $n = 4$. Numbers above the solid arcs with arrows indicate the corresponding transition probabilities. Dashed-lines with arrows connect the trajectories for different values of l .

correlation function $\mathcal{C}^{(k)} = \mathcal{C}_{j, j+1, \dots, j+k}^{(k)}$ of a consecutive sequence of arbitrary order k can be obtained exactly. One has that $\mathcal{C}^{(k)}$ obey the following recursion [9]:

$$\mathcal{C}^{(k)} = \sum_{p=0}^{k-1} \frac{(-1)^k 2^k}{(k+1)!} \mathcal{C}^{(k-1-p)}, \quad \mathcal{C}^{(0)} = 1, \quad (2.15)$$

which implies that $\mathcal{C}^{(k)}$ are related to the tangent numbers [15], and are given explicitly by

$$\mathcal{C}^{(k)} = \frac{(-1)^k 2^{k+2} (2^{k+2} - 1)}{(k+2)!} B_{k+2}, \quad (2.16)$$

where B_k are the Bernoulli numbers. Note that since the Bernoulli numbers equal zero for k odd, correlation functions of odd order vanish, i.e., $\mathcal{C}^{(2k+1)} \equiv 0$. Eq.(2.16) also implies that for $k \gg 1$, the k -point correlation functions $\mathcal{C}^{(k)}$ (k is even) decay as $\mathcal{C}^{(k)} \sim 2(-1)^{k/2} (2/\pi)^{k+2}$.

2.3 Probability Distribution of $X_l^{(n)}$ for $1 < n$.

To determine the structure of excursions $X_l^{(n)}$ of the PGRW we adapt a method proposed by Hammersley [14] in his analysis of the expected length of the longest increasing subsequence. The basic idea behind this approach is to build recursively an auxiliary *Markovian* stochastic process Y_l , which is distributed exactly as $X_l^{(n)}$.

At each time step l , we define a real valued random variable x_{l+l} , uniformly distributed in $[0, 1]$. Further on, we consider a random walk on a one-dimensional lattice of integers

whose trajectory Y_l is constructed according to the following step-by-step process: at each time moment l a point-like particle is created at position x_{l+1} . If $x_{l+1} > x_l$, a walker is moved one step to the right; otherwise, it is moved one step to the left. The trajectory Y_l is then given by $Y_l = \sum_{k=1}^l \text{sgn}(x_{k+1} - x_k)$, where $\text{sgn}(x)$ is defined in Eq.(2.2).

We note that the joint process (x_{l+1}, Y_l) , and therefore Y_l , are *Markovian* since they depend only on (x_l, Y_{l-1}) . Note also that Y_l is the sum of *correlated* random variables; hence, one has to be cautious when applying central limit theorems. A central limit theorem indeed holds for the Markovian process Y_l , but the summation rule for the variance is not valid.

Two following results have been proven in Ref. [9] concerning the relation between this recursively constructed Markovian process Y_l and the PGRW:

- (a) the probability $P(Y_l = X)$ that the trajectory Y_l of the auxiliary process appears at site X at time moment l is equal to the probability $\mathcal{P}_l(X) = P(X_l^{(l)} = X)$ that the end-point X_l of the PGRW trajectory $X_l^{(l)}$ generated by a given permutation of $[l+1]$ appears at site X .
- (b) the probability $P(X_l^{(n)} = X)$ that the PGRW trajectory $X_l^{(n)}$ at intermediate time moment $l = 1, 2, 3, \dots, n$ appears at site X is equal to the probability $P(Y_l = X)$ that the trajectory of the auxiliary process Y_l appears at time moment l at site X . In other words, one has

$$P(Y_l = X) = P(X_l^{(n)} = X) = \mathcal{P}_l(X) = \frac{[1 + (-1)^{X+l}]}{2(l+1)!} \left\langle \frac{l+1}{\frac{X+l}{2}} \right\rangle. \quad (2.17)$$

Note that distribution of any intermediate point $X_l^{(n)}$ of the PGRW trajectory generated by permutations of a sequence of length $n+1$ depends on l but is independent of n .

2.4 Probability Measure of PGRW Trajectories.

The equivalence of the processes Y_l and $X_l^{(n)}$ allows to determine the probability measure of any given trajectory. We note that in the permutation language, this problem amounts to the calculation of the number of permutations with a given rise-and-descent sequence and has been already discussed using an elaborated combinatorial approach in Refs. [18,16,17]. A novel solution of this problem has been proposed in Ref. [9], which expressed the probability measure of any given PGRW trajectory (or of some part of it) as a chain of iterated integrals.

Consider a given rise-and-descent sequence $\alpha(k)$ of length k of the form:

$$\alpha(k) = \begin{pmatrix} 1 & 2 & 3 & \dots & k \\ a_1 & a_2 & a_3 & \dots & a_k \end{pmatrix}.$$

where a_l can take either of two symbolic values — \uparrow or \downarrow . Consequently, the first line in the table is the running index l which indicates position along the permutation, while the second line shows what we have at this position - a rise or a descent. Assign next to each symbol at position l an integral operator; $I_l(\uparrow)$ for a rise (\uparrow) and $I_l(\downarrow)$ for a descent (\downarrow):

$$\hat{I}_l(\uparrow) = \int_{x_{l-1}}^1 dx_l \quad \text{and} \quad \hat{I}_l(\downarrow) = \int_0^{x_{l-1}} dx_l. \quad (2.18)$$

Define next a characteristic polynomial $Q(x, \alpha(k))$ as an "ordered" product [9]:

$$Q(x, \alpha(k)) = \prod_{l=1}^k \hat{I}_l(a_l) \cdot 1, \quad (2.19)$$

where $a_l = \{\uparrow, \downarrow\}$ for $l = 1, \dots, k$. The probability measure $P(\alpha(k))$ of this given rise-and-descent sequence $\alpha(k)$ in the ensemble of all equally likely permutations is then given by

$$P(\alpha(k)) = \int_0^1 Q(x, \alpha(k)) dx. \quad (2.20)$$

It may be expedient to illustrate this formal consideration on a particular example. Let a given rise-and-descent sequence be of the form $\{\uparrow, \uparrow, \downarrow, \uparrow, \uparrow\}$. For this sequence, the characteristic polynomial $Q(x, \alpha(5))$ and the probability of this particular configuration obey

$$\begin{aligned} Q(x, \alpha(5)) &= \hat{I}_1(\uparrow) \hat{I}_2(\uparrow) \hat{I}_3(\downarrow) \hat{I}_4(\uparrow) \hat{I}_5(\uparrow) \cdot 1 = \int_x^1 dx_1 \int_{x_1}^1 dx_2 \int_0^{x_2} dx_3 \int_{x_3}^1 dx_4 \int_{x_4}^1 dx_5 \cdot 1 = \\ &= \frac{3}{40} - \frac{x}{8} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{120} \quad \text{and} \quad P(\alpha(5)) = \int_0^1 Q(x, \alpha(5)) dx = \frac{19}{720}. \end{aligned} \quad (2.21)$$

Another way to look on the problem of calculation of the measure of a given PGRF trajectory is to use the results of Niven determining the number of permutations with a given rise-and-descent sequence [17]. Following Niven, consider a fixed up-and-down arrow sequence $\alpha(k)$ of length k and denote by l_1, l_2, \dots, l_r the positions of downarrows (descents), r being the total number of downarrows along the sequence. A question now is to calculate the *number* $N(X_l^{(n)})$ of permutations generating a given up-and-down sequence (or, in our language, a given trajectory $X_l^{(n)}$). Combinatorial arguments show that $N(X_l^{(n)})$ equals the determinant of a matrix of order $r+1$ whose elements $a_{i,j}$ (where i stands for the row, while j - for the column) are binomial coefficients $\binom{l_i}{l_{j-1}}$, where $l_0 = 0$ and $l_{r+1} = k+1$ [17]. Consequently, an alternative expression for the probability $P(\alpha(k))$ may be written down as

$$P(\alpha(k)) = \frac{1}{(k+1)!} \det \begin{pmatrix} 1 & 1 & \binom{l_1}{l_2} & \binom{l_1}{l_3} & \dots & \binom{l_1}{l_r} \\ 1 & \binom{l_2}{l_1} & 1 & \binom{l_2}{l_3} & \dots & \binom{l_2}{l_r} \\ 1 & \binom{l_3}{l_1} & \binom{l_3}{l_2} & 1 & \dots & \binom{l_3}{l_r} \\ & & & \dots & & \\ 1 & \binom{l_{r+1}}{l_1} & \binom{l_{r+1}}{l_2} & \binom{l_{r+1}}{l_3} & \dots & \binom{l_{r+1}}{l_r} \end{pmatrix}. \quad (2.22)$$

One may readily verify that both Eq.(2.20) and Eq.(2.22) reproduce our earlier results determining the probabilities of different four step trajectories. Note also that the probability measure defined by Eq.(2.20) or Eq.(2.22) is not homogeneous, contrary to the measure of the standard Pólya walk.

2.5 Distribution of The Number of "U-Turns" of PGRW Trajectories.

In this subsection we study an important measure of how scrambled the PGRW trajectories are. This measure is the number \mathcal{N} of the "U-turns" of an n -step PGRW trajectory, i.e. the number of times when the walker changes the direction of its motion.

In the permutation language, each turn to the left (right), when the walker making a jump to the right (left) at time moment l jumps to the left (right) at the next time moment $l+1$ corresponds to a peak $\uparrow\downarrow$ (a through $\downarrow\uparrow$) of a given permutation π , i.e. a sequence $\pi_l < \pi_{l+1} > \pi_{l+2}$ ($\pi_l > \pi_{l+1} < \pi_{l+2}$). Consequently, the distribution function $\mathcal{P}(\mathcal{N}, n)$ of the number of the "U-turns" of the PGRW trajectory, (i.e. the probability that an n -step PGRW trajectory has exactly \mathcal{N} turns), is just the distribution of the sum of peaks and through.

Let \mathcal{N}_p and \mathcal{N}_t denote the number of peaks and troughs in a given random permutation $[n+1]$, respectively. These realization-dependent numbers can be written down as

$$\mathcal{N}_p = \frac{1}{4} \sum_{l=1}^{n-1} (1 + s_l) (1 - s_{l+1}), \quad \mathcal{N}_t = \frac{1}{4} \sum_{l=1}^{n-1} (1 - s_l) (1 + s_{l+1}), \quad (2.23)$$

such that total number \mathcal{N} of U-turns of the PGRW trajectory is given by

$$\mathcal{N} = \mathcal{N}_p + \mathcal{N}_t = \frac{1}{2} \sum_{l=1}^{n-1} (1 - s_l s_{l+1}) \quad (2.24)$$

One finds then that the characteristic function $\mathcal{Z}_n(k)$ of \mathcal{N} is a polynomial in $\tanh(ik/2)$ [9]:

$$\begin{aligned} \mathcal{Z}_n(k) &= \left\langle \exp(ik\mathcal{N}) \right\rangle = \exp\left(\frac{ik(n-1)}{2}\right) \left\langle \exp\left(-\frac{ik}{2} \sum_{l=1}^{n-1} s_l s_{l+1}\right) \right\rangle = \left(\frac{1+e^{ik}}{2}\right)^{n-1} \times \\ &\times \sum_{l=0}^{[n/2]} (-1)^l \left(\sum_{1m_1+2m_2+\dots+lm_l=l} \binom{n-2l+1}{\sum_{j=1}^l m_j} \frac{(\sum_{j=1}^l m_j)!}{m_1!m_2!\dots m_l!} \prod_{j=1}^l (\mathcal{C}^{(2j)})^{m_j} \right) \tanh^l\left(\frac{ik}{2}\right), \end{aligned}$$

where $\mathcal{C}^{(2j)}$ obey Eq.(2.16). The generating function of $\mathcal{Z}_n(k)$ is given by

$$\begin{aligned} \mathcal{Z}(k, z) &= \sum_{n=2}^{\infty} \mathcal{Z}_n(k) z^n = -\frac{4}{(1+e^{ik})^2 z} - \frac{2}{1+e^{ik}} - z + \\ &+ \frac{4}{(1+e^{ik})^2 z} \left[1 - \frac{(1+e^{ik})z}{2} \sum_{j=0}^{\infty} \mathcal{C}^{(2j)} \left(\frac{(1-e^{2ik})z^2}{4} \right)^j \right]^{-1} = \\ &= \frac{4}{(1+e^{ik})^2 z} \left[\left(\frac{1-e^{ik}}{1+e^{ik}} \right)^{1/2} \coth\left((1-e^{2ik})^{1/2} \frac{z}{2} \right) - 1 \right]^{-1} - \frac{2}{1+e^{ik}} - z \end{aligned}$$

Turning next to the limit $z \rightarrow 1^-$ and inverting $\mathcal{Z}(k, z)$ with respect to k and z , we find that in the asymptotic limit $n \rightarrow \infty$, the distribution function $\mathcal{P}(\mathcal{N}, n)$ of the number of the "U-turns" in the PGRW trajectory converges to a normal distribution:

$$\mathcal{P}(\mathcal{N}, n) \sim \frac{3}{4} \left(\frac{5}{\pi n} \right)^{1/2} \exp\left(-\frac{45 \left(\mathcal{N} - \frac{2}{3}n \right)^2}{16n} \right), \quad (2.25)$$

with mean value $2n/3$ and variance $\sigma^2 = 8n/45$.

2.6 Diffusion Limit

Consider finally a continuous space and time version of the PGRW in the diffusion limit. In Ref. [9], the following approach has been developed:

Define first the polynomial: $V^{(l)}(x, Y) = \sum_{Y_l=Y} Q(x, \alpha(l))$, where the polynomial $Q(x, \alpha(l))$ has been determined in Eq.(2.19) and the sum extends over all l -steps trajectories Y_l starting at zero and ending at the fixed point Y . Note next that one has $\mathcal{P}_l(Y) = \langle V^{(l)}(x, Y) \rangle_{\{x_l\}}$. Now,

for the polynomials $V^{(l)}(x, Y)$ one obtains, by counting all possible trajectories Y_l starting from zero and ending at the fixed point Y , the following "evolution" equation:

$$V^{(l+1)}(x, Y) = \hat{I}_l(\uparrow) \cdot V^{(l)}(x, Y - 1) + \hat{I}_l(\downarrow) \cdot V^{(l)}(x, Y + 1). \quad (2.26)$$

Taking next advantage of the established equivalence between the processes Y_l and $X_l^{(n)}$, we can rewrite the last equation, upon averaging it over the distribution of variables $\{x_l\}$, as

$$\mathcal{P}_{l+1}(Y) = \frac{(l - Y + 4)}{2(l + 1)} \mathcal{P}_l(Y - 1) + \frac{(l + Y)}{2(l + 1)} \mathcal{P}_l(Y + 1), \quad (2.27)$$

which represents the desired evolution equation for $\mathcal{P}_l(Y)$ in discrete space and time.

We turn next to the diffusion limit. Introducing $y = aY$ and $t = \tau l$ variables, where a and τ define characteristic space and time scales, we turn to the limit $a, \tau \rightarrow 0$, supposing that the ratio a^2/τ remains fixed and determines the diffusion coefficient $D_0 = a^2/2\tau$. In this limit, Eq.(2.27) becomes

$$\frac{\partial}{\partial t} \mathcal{P}(y, t) = \frac{\partial}{\partial y} \left(\frac{y}{t} \mathcal{P}(y, t) \right) + D_0 \frac{\partial^2}{\partial y^2} \mathcal{P}(y, t) \quad (2.28)$$

Note that the resulting continuous space and time equation is of the Fokker-Planck type; it has a constant diffusion coefficient and a negative drift term which, similarly to the Ornstein-Uhlenbeck process, grows linearly with y , but the amplitude of the drift decays in proportion to the first inverse power of time, which signifies that the process Y_l eventually delocalizes. The Green's function solution of Eq.(2.28), remarkably, is a normal distribution

$$\mathcal{P}(y, t) = \sqrt{\frac{3}{4\pi D_0 t}} \exp \left(-\frac{3y^2}{4D_0 t} \right), \quad (2.29)$$

which is consistent with the large- n limit of the discrete process derived in Eq.(2.8).

3 RANDOM SURFACES GENERATED BY RANDOM PERMUTATIONS.

Consider a one-dimensional lattice containing L sites on which we distribute at random numbers drawn from the set $1, 2, 3, \dots, L$, (see Fig.2). Suppose that a number appearing at the site j determines the local height of the surface. Further on, we denote a local "peak" as a site j the number at which exceeds the numbers appearing at two neighboring sites. Generalization to two-dimensional square lattice with $L = m \times m$, (m is an integer) sites is straightforward (see Fig.2): the only difference here is that we call as local surface peaks such sites j the numbers at which are greater than numbers appearing at four adjacent sites.

For these models, our goal is to evaluate the probability $P(M, L)$ that the surface created in such a way has exactly M peaks on a lattice containing L sites [10]. In one dimension this can be done exactly and provides also a distribution function of the number of right U-turns of the PGRW trajectories. Here we are also able to calculate the "correlation function" $p(l)$ defining the conditional probability that two peaks are separated by the interval l under the condition that the interval l does not contain other peaks. In 2D, determining exactly first three cumulants of $P(M, L)$ we define its asymptotic form using expansion in the Edgeworth series [11] and show that it converges to a normal distribution as $L \rightarrow \infty$ [10]. For 2D model, we will also discuss some surprising cooperative behavior of peaks.

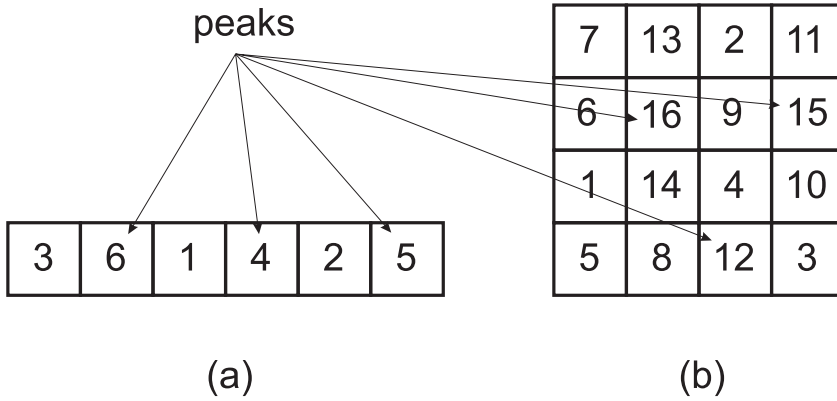


Fig. 2. (a) One-dimensional ($L = 6$) and (b) two-dimensional square ($L = 4 \times 4$) lattices with periodic boundary conditions on sites of which we distribute numbers drawn from the set $1, 2, 3, \dots, L$. These numbers determine the local heights of the surface. The sites with numbers bigger than those appearing on the neighboring sites are referred to as the "peaks".

3.1 Probability Distribution $P(M, L)$ in 1D Case

In one dimension, the probability $P(M, L)$ of having M surface peaks on a lattice containing L sites can be calculated exactly through its generating function [10]:

$$W(s, L) = L! \sum_{M=0}^{\infty} s^{M+1} P(M, L), \quad (3.1)$$

where

$$W(s, L) = \left(\frac{s}{1 - \sqrt{1-s}} \right)^{L+1} \sum_{M=0}^{\infty} \left(\frac{(1 - \sqrt{1-s})^2}{s} \right)^{M+1} \left\langle \begin{matrix} L \\ M-1 \end{matrix} \right\rangle. \quad (3.2)$$

Inverting (3.1), we find the following exact expression for $P(M, L)$:

$$P(M, L) = \frac{2^{L+2}}{L!} \sum_{l=1}^M (-1)^{M-l} \left(\frac{\frac{L+1}{2}}{M-l} \right) \sum_{m=1}^l \frac{m^{L+1}}{(l-m)!(m+l)!} \quad (3.3)$$

One finds then that in the limit $L \rightarrow \infty$, the probability $P(M, L)$ converges to a normal distribution:

$$P(M, L) \sim \frac{3}{2} \sqrt{\frac{5}{\pi L}} \exp \left\{ -\frac{45(M - \frac{1}{3}L)^2}{4L} \right\} \quad (3.4)$$

with mean $L/3$ and variance $\sigma^2 = 2L/45$. Note that this result is consistent with the distribution of the number of U-turns of the PGRW trajectories, Eq.(2.25).

3.2 Conditional Probability $p(l)$ of Two Peaks Separated by Distance l

We aim now at evaluating the conditional probability $p(l)$ of having two peaks separated by a distance l , under the condition that there are no peaks (i.e. sequences $\uparrow \downarrow$) on the interval

between these peaks. Using the approach put forth in Section 2.4, we have that this probability obeys

$$p(l) = \int_0^1 dx \sum Q(x, \alpha), \quad (3.5)$$

where the sum is taken over all possible peak-avoiding rise-and-descent patterns of length l between of two peaks, while $Q(x, \alpha)$ denote the Q -polynomials corresponding to each given configuration.

There are several possible peak-avoiding rise-and-descent sequences contributing to such a probability. These sequences are depicted in Fig.3.

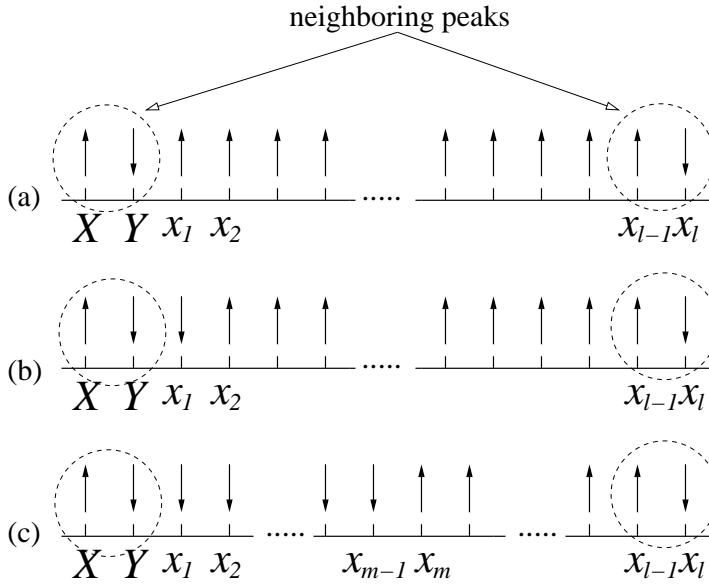


Fig. 3. Rise-and-descent patterns contributing to the conditional probability of having two closest peaks at distance l apart from each other. Configuration (a) has only one descent between two peaks. Configuration (b) has two descents following the first peak, i.e. a "through" at x_1 , and (c) presents a generalization of (b) over configurations having m descents, ($m = 1, 2, \dots, l$), after the first peak which are followed by $l - m$ rises, i.e. a "through" at x_{m-1} .

The Q -polynomial associated with the general configuration (c) in Fig.3 is given by

$$Q^c(x) = \int_x^1 dX \int_0^X dY \int_0^Y dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{m-2}} dx_{m-1} \int_{x_{m-1}}^1 dx_m \dots \int_{x_{l-2}}^1 dx_{l-1} \int_0^{x_{l-1}} dx_l \quad (3.6)$$

Calculating this integral recursively and summing over m , $m = 1, 2, \dots, l$, we find that the generating function of the probability $p(l)$ obeys

$$\begin{aligned} F(z) = \sum_{l=2} p(l) z^l &= \frac{(z-1)^2}{2z^3} e^{2z} + \left(\frac{1}{3} + \frac{1}{2z} - \frac{1}{2z^3} \right) = \\ &= \frac{2}{15} z^2 + \frac{1}{9} z^3 + \frac{2}{35} z^4 + \frac{1}{45} z^5 + \frac{4}{567} z^6 + \dots \end{aligned} \quad (3.7)$$

Inverting the latter expression, we find that $p(l)$ is determined explicitly by

$$p(l) = \frac{1}{2} \left(\frac{2^{l+1}}{(l+1)!} - 2 \frac{2^{l+2}}{(l+2)!} + \frac{2^{l+3}}{(l+3)!} \right) = 2^l \frac{(l-1)(l+2)}{(l+3)!}, \quad (3.8)$$

Note also that $p(l)$ in eq.(3.8) coincides with the distribution function of the distance between two "weak" bonds obtained by Derrida and Gardner [19] in their analysis of the number of metastable states in a one-dimensional Ising spin glass at zero temperature.

3.3 Probability Distribution $P(M, L)$ in 2D Case

We turn now to calculation of the probability $P(M, L)$ of having M peaks on a two-dimensional square lattice containing $m \times m = L$ sites. We linearly order lattice sites by index j , $j = 1, 2, \dots, L$, in the same way as an electron beam highlights the TV screen. Note that in such a representation a site j is a peak if and only if π_j is simultaneously larger than $\pi_{j-1}, \pi_{j+1}, \pi_{j-m}$ and π_{j+m} .

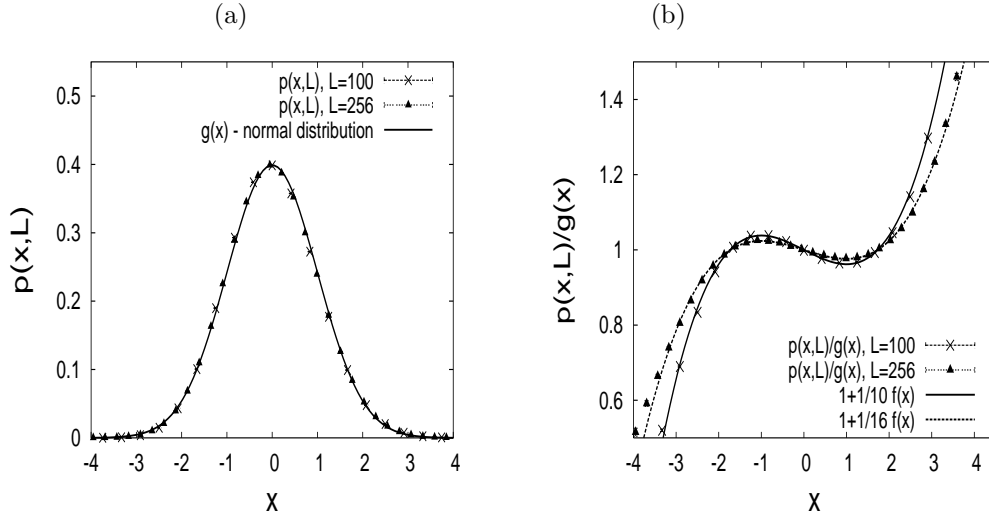


Fig. 4. (a) Results of numerical simulation of $p(x, L)$ for $L = 100, 256$: (a) comparison of $p(x, L)$ with the Gaussian function $g(x)$; (b) comparison of the function $p(x, L)/g(x)$ with $1 + f(x)/\sqrt{L}$.

Using the approach devised in Section 2.4, we find the first cumulant of $P(M, L)$:

$$\kappa_1^{2D} = \sum_j \int_0^1 \dots \int_0^1 \prod_k dx_k \theta(x_j - x_{j-1}) \theta(x_j - x_{j+1}) \theta(x_j - x_{j+m}) \theta(x_j - x_{j-m}) = \frac{1}{5} L \quad (3.9)$$

where the summation over "j" and "k" extends over all lattice sites, i.e. $j, k = 1, 2, 3, \dots, L$. Further on, we find that the second cumulant is given by

$$\begin{aligned} \kappa_2^{2D} &= \sum_j \sum_i \int_0^1 \dots \int_0^1 \prod_k dx_k \theta(x_j - x_{j-1}) \theta(x_j - x_{j+1}) \theta(x_j - x_{j+m}) \theta(x_j - x_{j-m}) \times \\ &\times \theta(x_i - x_{i-1}) \theta(x_i - x_{i+1}) \theta(x_i - x_{i+m}) \theta(x_i - x_{i-m}) - \frac{1}{25} L^2 = \frac{13}{225} L \end{aligned} \quad (3.10)$$

Calculation of the third cumulant is more involved [10] and yields

$$\kappa_3^{2D} = \frac{512}{32175}L. \quad (3.11)$$

As a matter of fact, we can show that cumulants of any order grow in proportion to the first power of L . Introducing next the normalized deviation, $x = (M - \kappa_1^{2D})/\sqrt{\kappa_2^{2D}}$, we seek the normalized probability distribution $p(x, L) = P(\kappa_1^{2D} + x\sqrt{\kappa_2^{2D}}, L)$ expanding it in the Edgeworth series (cumulant expansion) [11]. We find then that $p(x, L)$ obeys

$$p(x, L) \simeq g(x) \left(1 + \frac{1}{\sqrt{L}}f(x) + o\left(\frac{1}{\sqrt{L}}\right) \right), \quad (3.12)$$

where $g(x)$ is the Gaussian function $g(x) = \exp(-x^2/2)/\sqrt{2\pi}$ and $f(x)$ is given by

$$f(x) = \sqrt{L} \frac{\kappa_3^{2D}}{(\kappa_2^{2D})^{3/2}} \frac{1}{6}(x^3 - 3x) = \frac{512}{32175} \left(\frac{225}{13} \right)^{3/2} \frac{1}{6}(x^3 - 3x). \quad (3.13)$$

Note that $f(x)$ is independent of L and correction terms in Eq.(3.12) decay faster than $1/\sqrt{L}$.

To check the asymptotical result in Eq.(3.12), we have performed numerical simulations for the discrete 2D permutation-generated model with periodic boundary conditions and have computed the distribution function $p(x, L)$ numerically. In Fig.4a we present numerical data for $p(x, L)$ and compare it against the Gaussian function $g(x)$ for system sizes $L = 100$ and 256 . Furthermore, in Fig.4b we plot the ratio $p(x, L)/g(x)$ as the function of x . One notices that the deviation of the numerically computed function $p(x, L)$ from the Gaussian function $g(x)$ is indeed very small. Moreover, the difference between the normalized probability distribution function $p(x, L)$ and the Gaussian function $g(x)$ is smaller the larger L is.

3.4 Gas, Liquid and Solid of Peaks

In this last subsection we discuss a surprising collective behavior of peaks. Consider a somewhat exotic "statistical physics" model with the partition function:

$$Z = \sum_{\text{all permutations}} z^M, \quad (3.14)$$

where z is a fugacity and M determines the number of peaks for a given distribution of natural numbers $1, 2, 3, \dots, L$ on a lattice containing L sites. In one- and two-dimensions

$$\begin{aligned} M &= \sum_j \theta(\pi_j - \pi_{j-1})\theta(\pi_j - \pi_{j+1}) \\ M &= \sum_j \theta(\pi_j - \pi_{j-1})\theta(\pi_j - \pi_{j+1})\theta(\pi_j - \pi_{j+m})\theta(\pi_j - \pi_{j-m}) \end{aligned} \quad (3.15)$$

respectively. A striking feature of this model in two-dimensions, as communicated to us by B.Derrida who analyzed a similar partition function numerically, is that it undergoes a phase transition at $z \approx \exp(2.2)$ [20]!

Below we present some arguments why such a transition may indeed take place in two-dimensions: Note first that peaks, by definition, can not simultaneously occupy nearest-neighbor lattice sites. It means that if we view the peaks as some fictitious particles, they

One-Dimensional System

$$P(\text{---}\bullet\text{---}) = 1/3$$

$$P(\text{---}\bullet\bullet\text{---}) = 0$$

$$P(\text{---}\bullet\quad\bullet\text{---}) = 2/15 > 1/3 \times 1/3$$

$$P(\text{---}\bullet\quad\quad\bullet\text{---}) = 1/3 \times 1/3$$

Two-Dimensional System

$$P(\begin{array}{|c|c|c|} \hline & \bullet & \\ \hline & & \\ \hline \end{array}) = 1/5$$

$$P(\begin{array}{|c|c|c|} \hline & \bullet & \bullet \\ \hline & & \\ \hline \end{array}) = 0$$

$$P(\begin{array}{|c|c|c|} \hline & \bullet & \\ \hline & & \bullet \\ \hline \end{array}) = 1/25$$

$$P(\begin{array}{|c|c|c|} \hline & \bullet & \bullet \\ \hline & & \\ \hline \end{array}) = 2/45 > 1/25$$

$$P(\begin{array}{|c|c|c|} \hline & & \bullet \\ \hline & \bullet & \\ \hline \end{array}) = 1/20 > 1/25$$

Fig. 5. Probabilities of having a single peak and a pair of isolated peaks some distance apart from each other on one- and two-dimensional lattices. Filled circles denote peaks - sites with numbers exceeding numbers appearing on nearest-neighboring sites. First column presents probabilities of several configurations in 1d, while the second one describes the probabilities of a few possible configurations appearing in two dimensions.

are not point particles occupying just a single site but rather "hard-squares". In principle, this is already enough to produce a phase transition, but at somewhat higher values of z [21], since here a cost of each particle is higher requiring a particular arrangement of numbers on five lattice sites.

Still striking, it appears that peaks experience short-range "attractive" interactions. Note that a probability of having a single isolated peak is equal to $1/5$. The probability of having two peaks separated by distance exceeding $l = 2$ is $1/25 = 1/5^2$, which means that at such distance the peak do not feel each other. On the other hand, the probability of having two peaks at next-nearest-neighboring sites or in corners of each lattice cell, is greater than $1/25$, which means that peaks effectively "attract" each other! In Fig.4 we summarize the results for the probabilities of several simple configurations of peaks.

This "attractive" force between peaks can be thought off as a sort of a depletion force acting between colloidal particles: by definition, each peak is a site with a number which is greater than numbers on four neighboring sites. When two peaks are next-nearest-neighboring sites, they have a "common" site with a number which is restricted to be less than the least of the numbers on the peak sites, i.e. may attain less values than numbers on the sites around an isolated peak. This effect is even more pronounced for peaks occupying sites in the corners of a lattice cell - here these two peaks have two "common" numbers and each of them has to be less than the least of them.

Note finally that for clusters containing several peaks, the presence of such common sites

makes the interactions between peaks non-additive and dependent on the specific features of the cluster's geometry. Thus it is not evident at all whether the phase transition is in the Ising universality class.

4 ACKNOWLEDGMENTS

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